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Hamiltonian and quasi-Hamiltonian structures associated with semi-direct sums of Lie algebras

Wen-Xiu Ma[1] **and Min Chen**[2]

[1] Department of Mathematics, University of South Florida, Tampa, FL 33620-5700, USA
[2] Department of Applied Mathematics, Donghua University, Shanghai 200051, People's Republic of China

E-mail: mawx@math.usf.edu

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Abstract

The trace variational identity is generalized to zero curvature equations associated with non-semi-simple Lie algebras or, equivalently, Lie algebras possessing degenerate Killing forms. An application of the resulting generalized variational identity to a class of semi-direct sums of Lie algebras in the AKNS case furnishes Hamiltonian and quasi-Hamiltonian structures of the associated integrable couplings. Three examples of integrable couplings for the AKNS hierarchy are presented: one Hamiltonian and two quasi-Hamiltonian.

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1. Introduction

An approach to integrable couplings (see [1, 2] for definition) was recently proposed on the basis of semi-direct sums of Lie algebras [3, 4]. The presented examples of continuous integrable couplings [3] and discrete integrable couplings [4] show interesting mathematical structures behind integrable equations. The related theory helps us work towards completely classifying integrable equations from an algebraic point of view, because any Lie algebra has a semi-direct sum structure of a solvable Lie algebra and a semi-simple Lie algebra [5, 6].

Let G be a matrix loop algebra. We assume that a pair of matrix spectral problems

$$\begin{cases} \phi_x = U\phi = U(u, \lambda)\phi, \\ \phi_t = V\phi = V\left(u, u_x, \ldots, \dfrac{\partial^{m_0} u}{\partial x^{m_0}}; \lambda\right)\phi, \end{cases} \tag{1.1}$$

where ϕ_x and ϕ_t denote the derivatives with respect to x and t, $U, V \in G$ are a Lax pair, λ is a spectral parameter and m_0 is a natural number indicating the differential order, determines [7, 8] an integrable equation (1.2)

$$u_t = K(u), \tag{1.2}$$

through their isospectral (i.e., $\lambda_t = 0$) compatibility condition (i.e., zero curvature equation):

$$U_t - V_x + [U, V] = 0.$$

This means that a triple (U, V, K) satisfies

$$U'(u)[K] - V_x + [U, V] = 0, \qquad \text{where} \quad U'(u)[K] = \left.\frac{\partial}{\partial \varepsilon}\right|_{\varepsilon=0} U(u + \varepsilon K). \tag{1.3}$$

There exists a Lie algebraic structure for such triples [9].

To generate integrable couplings of equation (1.2), take a semi-direct sum of G with another matrix loop algebra G_c as introduced in [3]:

$$\bar{G} = G \Subset G_c. \tag{1.4}$$

The notion of semi-direct sums means that G and G_c satisfy

$$[G, G_c] \subseteq G_c,$$

where $[G, G_c] = \{[A, B] | A \in G, B \in G_c\}$. Obviously, G_c is an ideal Lie sub-algebra of $\bar{G}$. The subscript c indicates a contribution to the construction of integrable couplings. Then, choose a pair of enlarged matrix spectral problems

$$\begin{cases} \bar{\phi}_x = \bar{U}\bar{\phi} = \bar{U}(\bar{u}, \lambda)\bar{\phi}, \\ \bar{\phi}_t = \bar{V}\bar{\phi} = \bar{V}\left(\bar{u}, \bar{u}_x, \ldots, \dfrac{\partial^{m_0}\bar{u}}{\partial x^{m_0}}; \lambda\right)\bar{\phi}, \end{cases} \tag{1.5}$$

where the enlarged Lax pairs are given by

$$\bar{U} = U + U_c, \qquad \bar{V} = V + V_c, \qquad U_c, V_c \in G_c. \tag{1.6}$$

Obviously, under equation (1.2), the corresponding enlarged zero curvature equation

$$\bar{U}_t - \bar{V}_x + [\bar{U}, \bar{V}] = 0$$

is equivalent to

$$\begin{cases} U_t - V_x + [U, V] = 0, \\ U_{c,t} - V_{c,x} + [U, V_c] + [U_c, V] + [U_c, V_c] = 0. \end{cases} \tag{1.7}$$

The first equation here exactly presents equation (1.2) and, thus, it provides a coupling system for equation (1.2). The analysis given here shows the basic idea of constructing continuous integrable couplings by using semi-direct sums of Lie algebras, proposed in [3] (see [3] for more information).

A class of semi-direct sums of matrix loop algebras presented in [3] is as follows:

$$\bar{G} = G \Subset G_c, \qquad G = \{\mathrm{diag}(\underbrace{A, \ldots, A}_{\nu+1})\}, \qquad G_c = \left\{ \begin{pmatrix} 0 & B_1 & \cdots & B_\nu \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & B_1 \\ 0 & \cdots & \cdots & 0 \end{pmatrix} \right\}, \tag{1.8}$$

where the matrices $A, B_i, 1 \leqslant i \leqslant \nu$, are the arbitrary matrices of the same size as U. Note that the Lax matrices, $\mathrm{diag}(\underbrace{U, \ldots, U}_{\nu+1})$ and $\mathrm{diag}(\underbrace{V, \ldots, V}_{\nu+1})$, generate the same equation as U and V. Therefore, new enlarged Lax matrices $\bar{U}$ and $\bar{V}$ in $\bar{G} = G \Subset G_c$ can be chosen as

$$\bar{U} = \begin{pmatrix} U & U_{a_1} & \cdots & U_{a_\nu} \\ 0 & U & \ddots & \vdots \\ \vdots & \ddots & \ddots & U_{a_1} \\ 0 & \cdots & 0 & U \end{pmatrix}, \qquad \bar{V} = \begin{pmatrix} V & V_{a_1} & \cdots & V_{a_\nu} \\ 0 & V & \ddots & \vdots \\ \vdots & \ddots & \ddots & V_{a_1} \\ 0 & \cdots & 0 & V \end{pmatrix}. \tag{1.9}$$

Integrable couplings generated by perturbations are also associated with specific examples of Lax pairs in this class [1, 2].

One of interesting questions is how to generate Hamiltonian structures for integrable couplings associated with semi-direct sums of Lie algebras of the type (1.8). A bilinear form $\langle \cdot, \cdot \rangle$ on a vector space is said to be non-degenerate when if $\langle A, B\rangle = 0$ for all vectors A, then $B = 0$, and if $\langle A, B\rangle = 0$ for all vectors B, then $A = 0$. Since the Killing form is always degenerate on semi-direct sums of Lie algebras in (1.8), which are not semi-simple, the trace variational identity (see [10, 11]) can just be used to establish Hamiltonian structures for the original equations but does not involve any business related to the additional Lie algebra G_c.

In this paper, we would like to generalize the trace variational identity to semi-direct sums of Lie algebras to construct Hamiltonian structures of associated integrable couplings. The key point is that for a bilinear form $\langle \cdot, \cdot \rangle$ on a given Lie algebra g, we get rid of the invariance property

$$\langle \rho(A), \rho(B)\rangle = \langle A, B\rangle \tag{1.10}$$

under an isomorphism ρ of the Lie algebra g, but keep the symmetric property

$$\langle A, B\rangle = \langle B, A\rangle \tag{1.11}$$

and the invariance property under the Lie product

$$\langle A, [B, C]\rangle = \langle [A, B], C\rangle, \tag{1.12}$$

where $[\cdot, \cdot]$ is the Lie product of g.

We can have plenty of non-degenerate bilinear forms of the above type on a semi-direct sum of Lie algebras. In what follows, we would like to show that the generalized variational identity under such a non-degenerate bilinear form can hold and be used to present Hamiltonian structures of integrable equations associated with semi-direct sums of Lie algebras. An application to the AKNS case furnishes Hamiltonian and quasi-Hamiltonian structures of integrable couplings associated with semi-direct sums of Lie algebras of the type (1.8). Three examples of integrable couplings for the AKNS hierarchy are presented: one Hamiltonian and two quasi-Hamiltonian. This also ensures that the algorithm to enlarge integrable systems using semi-direct sums of Lie algebras in [3] is efficient in presenting integrable couplings possessing Hamiltonian structures.

2. A variational identity under general bilinear forms

For a given spectral matrix $U = U(u, \lambda) \in G$, where G is a matrix loop algebra, let us fix the proper ranks $\operatorname{rank}(\lambda)$ and $\operatorname{rank}(u)$ so that U has the same rank, or it is homogeneous in rank, i.e., we can define

$$\operatorname{rank}(U) = \operatorname{rank}\left(\frac{\partial}{\partial x}\right) = \beta = \text{const.} \tag{2.1}$$

We assume that if two solutions V_1 and V_2 of the stationary zero curvature equation

$$V_x = [U, V] \tag{2.2}$$

possess the same rank, then they are linearly dependent of each other:

$$V_1 = \gamma V_2, \qquad \gamma = \text{const.} \tag{2.3}$$

This condition has also been required in deducing the standard trace variational identity [10, 11].

Associated with a non-degenerate bilinear form $\langle\cdot,\cdot\rangle$ on G with the symmetric property (1.11) and the invariance property (1.12), introduce a functional

$$W = \int (\langle V, U_\lambda\rangle + \langle \Lambda, V_x - [U, V]\rangle)\,\mathrm{d}x, \tag{2.4}$$

while U_λ denotes the partial derivative with respect to λ, and $V, \Lambda \in G$ are two matrices to be determined. The variational derivative $\nabla_A r \in G$ of a functional r with respect to $A \in G$ is defined by

$$\int \langle \nabla_A r, B\rangle\,\mathrm{d}x = \frac{\partial}{\partial \varepsilon} r(A + \varepsilon B)\Big|_{\varepsilon=0}, \qquad B \in G. \tag{2.5}$$

Obviously, based on the non-degenerate property of the bilinear form $\langle\cdot,\cdot\rangle$, we can have

$$\nabla_B \int \langle A, B\rangle\,\mathrm{d}x = A, \qquad \nabla_B \int \langle A, B_x\rangle\,\mathrm{d}x = -A_x.$$

Therefore, it follows from (1.11) and (1.12) that

$$\nabla_V W = U_\lambda - \Lambda_x + [U, \Lambda], \qquad \nabla_\Lambda W = V_x - [U, V]. \tag{2.6}$$

For the variational calculation of W, we require the following constrained conditions:

$$\nabla_V W = U_\lambda - \Lambda_x + [U, \Lambda] = 0, \tag{2.7}$$

$$\nabla_\Lambda W = V_x - [U, V] = 0, \tag{2.8}$$

which also imply that V and Λ are related to U and thus to the potential u. Based on (2.8), we have

$$\frac{\delta}{\delta u}\int \langle V, U_\lambda\rangle\,\mathrm{d}x = \frac{\delta W}{\delta u}, \tag{2.9}$$

where $\frac{\delta}{\delta u}$ is the variational derivative with respect to the potential u. In this formula, the dependence of u in V and U_λ needs to be considered in computing the left-hand side; while only the dependence of u in U needs to be considered in computing the right-hand side, due to both the above constrained conditions ((2.7) and (2.8)) and the property that if $\nabla_A r(A) = 0$, then $\frac{\delta}{\delta u} r(A(u)) = 0$. Thus, based on the invariance property (1.12), we have

$$\frac{\delta}{\delta u}\int \langle V, U_\lambda\rangle\,\mathrm{d}x = \frac{\delta W}{\delta u} = \left\langle V, \frac{\partial U_\lambda}{\partial u}\right\rangle + \left\langle [\Lambda, V], \frac{\partial U}{\partial u}\right\rangle. \tag{2.10}$$

Using (2.7), (2.8) and the Jacobi identity, we have

$$\begin{aligned}[\Lambda, V]_x &= [\Lambda_x, V] + [\Lambda, V_x] = [U_\lambda + [U, \Lambda], V] + [\Lambda, [U, V]] \\ &= [U_\lambda, V] + [V, [\Lambda, U]] + [\Lambda, [U, V]] = [U_\lambda, V] + [U, [\Lambda, V]];\end{aligned} \tag{2.11}$$

and from (2.8), we have

$$V_{\lambda,x} = V_{x,\lambda} = [U_\lambda, V] + [U, V_\lambda]. \tag{2.12}$$

Therefore, $Z := [\Lambda, V] - V_\lambda$ satisfies

$$Z_x = [U, Z].$$

By taking use of the uniqueness condition in (2.3) and $\mathrm{rank}(Z) = \mathrm{rank}(V_\lambda) = \mathrm{rank}\left(\frac{1}{\lambda}V\right)$, there exists a constant γ so that

$$[\Lambda, V] - V_\lambda = Z = \frac{\gamma}{\lambda} V, \tag{2.13}$$

because $\frac{1}{\lambda}V$ is a solution of (2.2). Finally, (2.10) can be expressed as

$$\begin{aligned}\frac{\delta}{\delta u}\int\langle V,U_\lambda\rangle\,\mathrm{d}x &= \left\langle V,\frac{\partial U_\lambda}{\partial u}\right\rangle+\left\langle V_\lambda,\frac{\partial U}{\partial u}\right\rangle+\frac{\gamma}{\lambda}\left\langle V,\frac{\partial U}{\partial u}\right\rangle\\ &= \frac{\partial}{\partial\lambda}\left\langle V,\frac{\partial U}{\partial u}\right\rangle+\left(\lambda^{-\gamma}\frac{\partial}{\partial\lambda}\lambda^{\gamma}\right)\left\langle V,\frac{\partial U}{\partial u}\right\rangle\\ &= \lambda^{-\gamma}\frac{\partial}{\partial\lambda}\lambda^{\gamma}\left\langle V,\frac{\partial U}{\partial u}\right\rangle.\end{aligned}\tag{2.14}$$

We summarize the above result in the following theorem.

Theorem (the variational identity under general bilinear forms)**.** *Let G be a matrix loop algebra, $U = U(u,\lambda) \in G$ be homogeneous in rank and $\langle\cdot,\cdot\rangle$ denote a non-degenerate symmetric bilinear form invariant under the matrix Lie product. Assume that the stationary zero curvature equation (2.2) has a unique solution $V \in G$ of a fixed rank up to a constant multiplier. Then for any solution $V \in G$ of (2.2), being homogeneous in rank, we have the following variational identity:*

$$\frac{\delta}{\delta u}\int\langle V,U_\lambda\rangle\,\mathrm{d}x=\lambda^{-\gamma}\frac{\partial}{\partial\lambda}\lambda^{\gamma}\left\langle V,\frac{\partial U}{\partial u}\right\rangle,\tag{2.15}$$

where γ is a constant.

In the above theorem, we did not require the invariance property under isomorphisms, defined in (1.10). Therefore, the variational identity (2.15) generalizes the standard trace variational identity in [10, 11], and it can be applied to both semi-simple and non-semi-simple Lie algebras.

Moreover, if we fix a basis of an involved Lie algebra and transform the spectral problem introduced in [12] into a matrix spectral problem of the type (1.1), we can easily see that the so-called quadratic-form identity in [12] is just a consequence of this generalized variational identity (2.15).

3. Constructing non-degenerate bilinear forms

To construct non-degenerate bilinear forms, we transform the semi-direct sum $\bar{G}$ of Lie algebras, defined by (1.8), into a vector form, but we consider the whole process only through one specific example.

Define the mapping

$$\delta:\bar{G}\to R^6,\qquad A\mapsto(a_1,a_2,a_3,a_4,a_5,a_6)^T,\qquad A=\begin{bmatrix}a_1&a_2&a_4&a_5\\a_3&-a_1&a_6&-a_4\\0&0&a_1&a_2\\0&0&a_3&-a_1\end{bmatrix}\in\bar{G}.\tag{3.1}$$

This mapping δ induces a Lie algebraic structure on $\mathbb{R}^6$, isomorphic to the matrix loop algebra $\bar{G}$. The corresponding commutator $[\cdot,\cdot]$ on $\mathbb{R}^6$ is given by

$$[a,b]^T=a^TR(b),\qquad a=(a_1,\ldots,a_6)^T,\qquad b=(b_1,\ldots,b_6)^T\in\mathbb{R}^6,\tag{3.2}$$

where

$$R(b) = \begin{bmatrix} 0 & 2b_2 & -2b_3 & 0 & 2b_5 & -2b_6 \\ b_3 & -2b_1 & 0 & b_6 & -2b_4 & 0 \\ -b_2 & 0 & 2b_1 & -b_5 & 0 & 2b_4 \\ 0 & 0 & 0 & 0 & 2b_2 & -2b_3 \\ 0 & 0 & 0 & b_3 & -2b_1 & 0 \\ 0 & 0 & 0 & -b_2 & 0 & 2b_1 \end{bmatrix}.$$

Any bilinear form on $\mathbb{R}^6$ can be defined by

$$\langle a, b\rangle = a^T F b, \tag{3.3}$$

where F is a constant matrix. The symmetric property $\langle a, b\rangle = \langle b, a\rangle$ requires that

$$F^T = F. \tag{3.4}$$

Under this symmetry condition, the invariance property under the Lie product

$$\langle a, [b, c]\rangle = \langle [a, b], c\rangle \tag{3.5}$$

requires that

$$F(R(b))^T = -R(b)F, \qquad \text{for all} \quad b \in \mathbb{R}^6 \tag{3.6}$$

noting $[b, c] = -[c, b]$ in (3.5). This matrix equation leads to a linear system of equations on the elements of F. Solving the resulting system yields

$$F = \begin{bmatrix} 2\eta_1 & 0 & 0 & 2\eta_2 & 0 & 0 \\ 0 & 0 & \eta_1 & 0 & 0 & \eta_2 \\ 0 & \eta_1 & 0 & 0 & \eta_2 & 0 \\ 2\eta_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \eta_2 & 0 & 0 & 0 \\ 0 & \eta_2 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{3.7}$$

where η_1 and η_2 are the arbitrary constants. Now, a bilinear form on the semi-direct sum $\bar{G}$ of Lie algebras, determined by (1.8), can be defined by

$$\begin{aligned} \langle A, B\rangle_{\bar{G}} &= \langle \delta^{-1}(A), \delta^{-1}(B)\rangle_{\mathbb{R}^6} = (a_1, \ldots, a_6)F(b_1, \ldots, b_6)^T \\ &= \eta_1(2a_1b_1 + a_2b_3 + a_3b_2) + \eta_2(2a_1b_4 + a_2b_6 + a_3b_5 + 2a_4b_1 + a_5b_3 + a_6b_2), \end{aligned} \tag{3.8}$$

where

$$A = \begin{bmatrix} a_1 & a_2 & a_4 & a_5 \\ a_3 & -a_1 & a_6 & -a_4 \\ 0 & 0 & a_1 & a_2 \\ 0 & 0 & a_3 & -a_1 \end{bmatrix}, \qquad B = \begin{bmatrix} b_1 & b_2 & b_4 & b_5 \\ b_3 & -b_1 & b_6 & -b_4 \\ 0 & 0 & b_1 & b_2 \\ 0 & 0 & b_3 & -b_1 \end{bmatrix}.$$

This bilinear form (3.8) is symmetric and invariant with the Lie product:

$$\langle A, B\rangle = \langle B, A\rangle, \qquad \langle A, [B, C]\rangle = \langle [A, B], C\rangle, \qquad A, B, C \in \bar{G}.$$

More importantly, the bilinear form (3.8) is non-degenerate if $\eta_2 \neq 0$. Thus, these kinds of bilinear forms can be used to establish Hamiltonian and quasi-Hamiltonian structures associated with semi-direct sums of Lie algebras, though they do not satisfy the invariance property (1.10) under isomorphisms (so, not of the Killing type, either).

We remark that the above construction procedure works for any other semi-direct sums of matrix Lie algebras. In the next section, we would like to shed light on the application process of the above bilinear form through three examples in the case of the AKNS hierarchy.

4. Applications

4.1. The AKNS hierarchy

Let us here recall the AKNS hierarchy [13]. The AKNS spectral problem is given by

$$\phi_x = U\phi, \qquad U = U(u,\lambda) = \begin{bmatrix} -\lambda & p \\ q & \lambda \end{bmatrix}, \qquad u = \begin{bmatrix} p \\ q \end{bmatrix}, \tag{4.1}$$

where p and q are the two dependent variables. Upon setting

$$W = \begin{bmatrix} a & b \\ c & -a \end{bmatrix} = \sum_{i\geqslant 0} W_i \lambda^{-i} = \sum_{i\geqslant 0} \begin{bmatrix} a_i & b_i \\ c_i & -a_i \end{bmatrix} \lambda^{-i}, \tag{4.2}$$

and choosing the initial data

$$a_0 = -1, \qquad b_0 = c_0 = 0,$$

we see that the stationary zero curvature equation $W_x = [U, W]$ generates

$$\begin{cases} b_{i+1} = -\frac{1}{2} b_{i,x} - p a_i, \\ c_{i+1} = \frac{1}{2} c_{i,x} - q a_i, \\ a_{i+1,x} = p c_{i+1} - q b_{i+1}, \end{cases} \tag{4.3}$$

where $i \geqslant 0$. Assume $a_i|_{u=0} = b_i|_{u=0} = c_i|_{u=0} = 0, i \geqslant 1$, or equivalently select constants of integration to be zero. Then, the recursion relation (4.3) uniquely defines a series of sets of differential polynomial functions in u with respect to x. The first three sets are as follows:

$$\begin{cases} b_1 = p, c_1 = q, a_1 = 0; \\ b_2 = -\frac{1}{2} p_x, c_2 = \frac{1}{2} q_x, a_2 = \frac{1}{2} pq; \\ b_3 = \frac{1}{4} p_{xx} - \frac{1}{2} p^2 q, c_3 = \frac{1}{4} q_{xx} - \frac{1}{2} pq^2, a_3 = \frac{1}{4}(pq_x - p_x q). \end{cases}$$

The compatibility conditions of the matrix spectral problems,

$$\phi_x = U\phi, \quad \phi_t = V^{[m]}\phi, \quad V^{[m]} = (\lambda^m W)_+, \quad m \geqslant 0, \tag{4.4}$$

determine the AKNS hierarchy of soliton equations

$$u_{t_m} = \begin{bmatrix} p \\ q \end{bmatrix}_{t_m} = K_m = \begin{bmatrix} -2b_{m+1} \\ 2c_{m+1} \end{bmatrix} = \Phi^m \begin{bmatrix} -2p \\ 2q \end{bmatrix} = J\frac{\delta H_m}{\delta u}, \qquad m \geqslant 0, \tag{4.5}$$

where the Hamiltonian operator J, the hereditary recursion operator Φ and the Hamiltonian functions are defined by

$$J = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}, \qquad \Phi = \begin{bmatrix} -\frac{1}{2}\partial + p\partial^{-1}q & p\partial^{-1}p \\ -q\partial^{-1}q & \frac{1}{2}\partial - q\partial^{-1}p \end{bmatrix}, \qquad H_m = \int \frac{2a_{m+2}}{m+1}\,\mathrm{d}x, \tag{4.6}$$

where $\partial = \frac{\partial}{\partial x}$ and $m \geqslant 0$.

4.2. Enlarged AKNS equations

As in [3], introduce two Lie algebras of 4×4 matrices:

$$G = \left\{ \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} \middle| A \in \mathbb{R}[\lambda] \otimes sl(2) \right\}, \qquad G_c = \left\{ \begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix} \middle| B \in \mathbb{R}[\lambda] \otimes sl(2) \right\}, \tag{4.7}$$

where the loop algebra $\mathbb{R}[\lambda]\otimes sl(2)$ is defined by span$\{\lambda^n A|n\geqslant 0, A\in sl(2)\}$, and form a semi-direct sum $\bar{G}=G\Subset G_c$ of these two Lie algebras G and G_c. In this case, G_c is an Abelian ideal of $\bar{G}$. For the AKNS spectral problem (4.1), we write

$$U=U_0\lambda+U_1,\qquad U_0=\begin{bmatrix}-1&0\\0&1\end{bmatrix},\qquad U_1=\begin{bmatrix}0&p\\q&0\end{bmatrix},\tag{4.8}$$

and define the corresponding enlarged spectral matrix as follows:

$$\bar{U}=\bar{U}(\bar{u},\lambda)=\begin{bmatrix}U&U_a\\0&U\end{bmatrix}\in G\Subset G_c,\qquad U_a=U_a(v)=\begin{bmatrix}-v_1&v_2\\v_3&v_1\end{bmatrix},\tag{4.9}$$

where v_i, $1\leqslant i\leqslant 3$, are new dependent variables and

$$v=(v_1,v_2,v_3)^T,\qquad \bar{u}=(u^T,v^T)^T=(p,q,v_1,v_2,v_3)^T.\tag{4.10}$$

To solve the corresponding enlarged stationary zero curvature equation $\bar{W}_x=[\bar{U},\bar{W}]$, we set

$$\bar{W}=\begin{bmatrix}W&W_a\\0&W\end{bmatrix},\qquad W_a=W_a(\bar{u},\lambda)=\begin{bmatrix}e&f\\g&-e\end{bmatrix},\tag{4.11}$$

where W is a solution to $W_x=[U,W]$, defined by (4.2). Then, the enlarged stationary zero curvature equation becomes

$$W_{a,x}=[U,W_a]+[U_a,W].\tag{4.12}$$

This equation is equivalent to

$$\begin{cases}e_x=pg-qf+v_2c-v_3b,\\ f_x=-2\lambda f-2pe-2v_1b-2v_2a,\\ g_x=2qe+2\lambda g+2v_3a+2v_1c.\end{cases}$$

Trying a solution

$$e=\sum_{i\geqslant 0}e_i\lambda^{-i},\qquad f=\sum_{i\geqslant 0}f_i\lambda^{-i},\qquad g=\sum_{i\geqslant 0}g_i\lambda^{-i},$$

we obtain

$$\begin{cases}e_{i+1,x}=pg_{i+1}-qf_{i+1}+v_2c_{i+1}-v_3b_{i+1},\\ f_{i+1}=-\frac{1}{2}f_{i,x}-pe_i-v_1b_i-v_2a_i,\\ g_{i+1}=\frac{1}{2}g_{i,x}-qe_i-v_3a_i-v_1c_i,\end{cases}\tag{4.13}$$

where $i\geqslant 0$, upon setting

$$e_0=-1,\qquad f_0=g_0=0.\tag{4.14}$$

Assuming $e_i|_{u=0}=f_i|_{u=0}=g_i|_{u=0}=0$, $i\geqslant 1$, we see that all sets of functions e_i, f_i and g_i are uniquely determined. In particular, the first few sets are

$$\begin{cases}f_1=p+v_2,\ g_1=q+v_3,\ e_1=0;\\ f_2=-\frac{1}{2}(p+v_2)_x-v_1p,\ g_2=\frac{1}{2}(q+v_3)_x-v_1q,\ e_2=\frac{1}{2}pq+\frac{1}{2}v_3p+\frac{1}{2}v_2q;\\ f_3=\frac{1}{4}(p+v_2)_{xx}+\frac{1}{2}(v_1p)_x-\frac{1}{2}p(pq+v_3p+v_2q)+\frac{1}{2}v_1p_x-\frac{1}{2}v_2pq;\\ g_3=\frac{1}{4}(q+v_3)_{xx}-\frac{1}{2}(v_1q)_x-\frac{1}{2}(pq+v_3p+v_2q)q-\frac{1}{2}v_3pq-\frac{1}{2}v_1q_x;\\ e_3=\frac{1}{4}(pq_x-p_xq)+\frac{1}{4}(v_{3,x}p-p_xv_3)+\frac{1}{4}(v_2q_x-v_{2,x}q)-v_1pq.\end{cases}$$

Expand W_a as

$$W_a=\sum_{i\geqslant 0}W_{a,i}\lambda^{-i}\tag{4.15}$$

and then it follows from (4.12) that

$$(W_{a,i})_x = [U_0, W_{a,i+1}] + [U_1, W_{a,i}] + [U_a, W_i], \qquad i \geqslant 0. \tag{4.16}$$

Now, we define

$$\bar{V}^{[m]} = \begin{bmatrix} V^{[m]} & V_a^{[m]} \\ 0 & V^{[m]} \end{bmatrix} \in \bar{G}, \qquad V_a^{[m]} = (\lambda^m W_a)_+ + \Delta_{m,a}, \qquad m \geqslant 0, \tag{4.17}$$

where $V^{[m]}$ is defined as in (4.4), and choose $\Delta_{m,a}$ as

$$\Delta_{m,a} = \begin{bmatrix} -\delta_m & 0 \\ 0 & \delta_m \end{bmatrix}, \qquad m \geqslant 0, \tag{4.18}$$

where $\delta_m, m \geqslant 0$, is a series of functions to be determined. Then, the mth enlarged zero curvature equation

$$\bar{U}_{t_m} - (\bar{V}^{[m]})_x + [\bar{U}, \bar{V}^{[m]}] = 0$$

leads to

$$U_{a,t_m} - \left(V_a^{[m]}\right)_x + \left[U, V_a^{[m]}\right] + [U_a, V^{[m]}] = 0,$$

together with the mth AKNS equation in (4.5). Based on (4.16), this can be simplified to

$$U_{a,t_m} - (\Delta_{m,a})_x - [U_0, W_{a,m+1}] + [U_1, \Delta_{m,a}] = 0,$$

which gives rise to

$$v_{t_m} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}_{t_m} = S_m(u, v) = \begin{bmatrix} \delta_{m,x} \\ -2f_{m+1} - 2p\delta_m \\ 2g_{m+1} + 2q\delta_m \end{bmatrix}, \qquad m \geqslant 0, \tag{4.19}$$

where $v = (v_1, v_2, v_3)^T$ defined as in (4.10). Therefore, we obtain a hierarchy of coupling systems,

$$\bar{u}_{t_m} = \begin{bmatrix} u \\ v \end{bmatrix}_{t_m} = \bar{K}_m(u) = \begin{bmatrix} K_m(u) \\ S_m(u, v) \end{bmatrix}, \qquad m \geqslant 0, \tag{4.20}$$

for the AKNS hierarchy (4.5) (see [3] for more information).

To construct Hamiltonian and quasi-Hamiltonian integrable couplings by using the variational identity (2.15), we consider a non-degenerate bilinear form on $\bar{G} = G \Subset G_c$ defined by (3.8) under the case

$$\eta_1 = \eta_2 = 1. \tag{4.21}$$

Therefore, we have

$$\langle \bar{W}, \bar{U}_\lambda \rangle = -2a - 2e, \tag{4.22}$$

where $\bar{U}$ and $\bar{W}$ are defined by (4.9) and (4.11), respectively. This quantity will be used to generate Hamiltonian functions for specific integrable couplings in (4.20).

4.3. Hamiltonian integrable couplings

Let $v_1 = \alpha = \text{const}$, and so $\bar{u} = (p, q, v_2, v_3)^T$. In this case, we have

$$\left\langle \bar{V}, \frac{\partial \bar{U}}{\partial \bar{u}} \right\rangle = (c + g, b + f, c, b)^T. \tag{4.23}$$

Therefore, the variational identity (2.15) leads to

$$\frac{\delta}{\delta \bar{u}} \int (-2a - 2e)\,\mathrm{d}x = \lambda^{-\gamma} \frac{\partial}{\partial \lambda} \lambda^{\gamma} (c + g, b + f, c, b)^T,$$

which is equivalent to

$$\frac{\delta}{\delta \bar{u}} \int (-2a_{m+1} - 2e_{m+1})\,\mathrm{d}x = (-m + \gamma)(c_m + g_m, b_m + f_m, c_m, b_m)^T,$$

by comparing the coefficients of λ^{-m-1}, $m \geqslant 0$. Taking $m = 0$ yields the constant $\gamma = 0$. Therefore, we have

$$(c_m + g_m, b_m + f_m, c_m, b_m)^T = \frac{\delta}{\delta \bar{u}} \int \frac{2(a_{m+1} + e_{m+1})}{m}\,\mathrm{d}x, \qquad m \geqslant 1. \tag{4.24}$$

Let us now choose

$$\delta_m = 0, \qquad m \geqslant 0, \tag{4.25}$$

in (4.18). Then the reduced case of (4.19) becomes

$$v_{t_m} = \begin{bmatrix} v_2 \\ v_3 \end{bmatrix}_{t_m} = S_m(u, v) = \begin{bmatrix} -2f_{m+1} \\ 2g_{m+1} \end{bmatrix}, \qquad m \geqslant 0,$$

where $v = (v_2, v_3)^T$. Further, the reduced hierarchy of (4.20) reads

$$\bar{u}_{t_m} = \begin{bmatrix} u \\ v \end{bmatrix}_{t_m} = \bar{K}_m(\bar{u}) = \begin{bmatrix} K_m(u) \\ S_m(u, v) \end{bmatrix} = \begin{bmatrix} -2b_{m+1} \\ 2c_{m+1} \\ -2f_{m+1} \\ 2g_{m+1} \end{bmatrix}, \qquad m \geqslant 0. \tag{4.26}$$

The first nonlinear system of (4.26) is

$$\begin{cases} p_{t_2} = -\frac{1}{2}p_{xx} + p^2 q,\ q_{t_2} = \frac{1}{2}q_{xx} - pq^2; \\ v_{2,t_2} = -\frac{1}{2}(p + v_2)_{xx} - 2\alpha p_x + p(pq + v_3 p + v_2 q) + v_2 pq; \\ v_{3,t_2} = \frac{1}{2}(p + v_3)_{xx} - 2\alpha q_x - (pq + v_3 p + v_2 q)q - v_3 pq. \end{cases} \tag{4.27}$$

Since we have

$$(-2b_{m+1}, 2c_{m+1}, -2f_{m+1}, 2g_{m+1})^T = \bar{J}(c_{m+1} + g_{m+1}, b_{m+1} + f_{m+1}, c_{m+1}, b_{m+1})^T,$$

where $\bar{J}$ is a Hamiltonian operator

$$\bar{J} = \begin{bmatrix} 0 & 0 & 0 & -2 \\ 0 & 0 & 2 & 0 \\ 0 & -2 & 0 & 2 \\ 2 & 0 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & J \\ J & -J \end{bmatrix}, \tag{4.28}$$

it then follows that the enlarged hierarchy (4.26) possesses the following Hamiltonian structure:

$$\bar{u}_{t_m} = \bar{K}_m = \bar{J} \frac{\delta \bar{\mathcal{H}}_m}{\delta \bar{u}}, \qquad m \geqslant 0, \tag{4.29}$$

where $\bar{J}$ is the Hamiltonian operator defined by (4.28) and

$$\bar{\mathcal{H}}_m = \int \frac{2(a_{m+2} + e_{m+2})}{m + 1}\,\mathrm{d}x, \qquad m \geqslant 0. \tag{4.30}$$

It is easy to see that $\bar{\mathcal{H}}_0 = \int (pv_3 + qv_2 + 2pq)\,\mathrm{d}x$.

Based on the recursion relations (4.3) and (4.13), we find the following hereditary recursion operator of any system in (4.26) [3]:

$$\bar{\Phi} = \bar{\Phi}(\bar{u}) = \begin{bmatrix} \Phi(u) & 0 \\ \Phi_c(\bar{u}) - \alpha I_2 & \Phi(u) \end{bmatrix}, \tag{4.31}$$

where $\Phi(u)$ is given by (4.6), I_2 is the identity matrix of order 2 and $\Phi_c(\bar{u})$ is defined by

$$\Phi_c(\bar{u}) = \begin{bmatrix} v_2\partial^{-1}q + p\partial^{-1}v_3 & v_2\partial^{-1}p + p\partial^{-1}v_2 \\ -v_3\partial^{-1}q - q\partial^{-1}v_3 & -v_3\partial^{-1}p - q\partial^{-1}v_2 \end{bmatrix}. \tag{4.32}$$

Moreover, it is easy to see that $\bar{J}\bar{\Phi}^* = \bar{\Phi}\bar{J}$. A direct computation can show that $\bar{J}$ and $\bar{\Phi}\bar{J}$ constitute a Hamiltonian pair and so the enlarged hierarchy (4.26) possesses a bi-Hamiltonian structure like the AKNS hierarchy. It also can be shown that the recursion operator $\bar{\Phi}$ in (4.31) is hereditary for all values of α.

4.4. Quasi-Hamiltonian integrable couplings

In a general case where v_1 is a variable, we have

$$\left\langle \bar{V}, \frac{\partial \bar{U}}{\partial \bar{u}} \right\rangle = (c+g, b+f, -2a, c, b)^T, \tag{4.33}$$

where $\bar{u} = (p, q, v_1, v_2, v_3)^T$. Therefore, the variational identity (2.15) leads to

$$\frac{\delta}{\delta \bar{u}} \int (-2a - 2e)\,\mathrm{d}x = \lambda^{-\gamma}\frac{\partial}{\partial\lambda}\lambda^{\gamma}(c+g, b+f, -2a, c, b)^T,$$

which gives rise to

$$\frac{\delta}{\delta \bar{u}} \int (-2a_{m+1} - 2e_{m+1})\,\mathrm{d}x = (-m+\gamma)(c_m+g_m, b_m+f_m, -2a_m, c_m, b_m)^T,$$

by comparing the coefficients of λ^{-m-1}, $m \geqslant 0$. Similarly, taking $m = 0$ yields $\gamma = 0$. Thus, we obtain

$$(c_m+g_m, b_m+f_m, -2a_m, c_m, b_m)^T = \frac{\delta}{\delta \bar{u}} \int \frac{2(a_{m+1}+e_{m+1})}{m}\,\mathrm{d}x, \qquad m \geqslant 1. \tag{4.34}$$

Case of $\delta_m = a_{m+1}$. Let us first choose

$$\delta_m = a_{m+1}, \qquad m \geqslant 0, \tag{4.35}$$

in (4.18). Then from (4.19), we have

$$v_{t_m} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}_{t_m} = S_m(u,v) = \begin{bmatrix} a_{m+1,x} \\ -2f_{m+1} - 2pa_{m+1} \\ 2g_{m+1} + 2qa_{m+1} \end{bmatrix}, \qquad m \geqslant 0,$$

where $v = (v_1, v_2, v_3)^T$. Moreover, the enlarged hierarchy of coupling systems for the AKNS hierarchy (4.5), defined by (4.20), reads

$$\bar{u}_{t_m} = \begin{bmatrix} u \\ v \end{bmatrix}_{t_m} = \bar{K}_m(\bar{u}) = \begin{bmatrix} K_m(u) \\ S_m(u,v) \end{bmatrix} = \begin{bmatrix} -2b_{m+1} \\ 2c_{m+1} \\ a_{m+1,x} \\ -2f_{m+1} - 2pa_{m+1} \\ 2g_{m+1} + 2qa_{m+1} \end{bmatrix}, \qquad m \geqslant 0. \tag{4.36}$$

It is easy to see that

$$\begin{bmatrix} -2b_{m+1} \\ 2c_{m+1} \\ a_{m+1,x} \\ -2f_{m+1} - 2pa_{m+1} \\ 2g_{m+1} + 2qa_{m+1} \end{bmatrix} = \bar{J} \begin{bmatrix} c_{m+1} + g_{m+1} \\ b_{m+1} + f_{m+1} \\ -2a_{m+1} \\ c_{m+1} \\ b_{m+1} \end{bmatrix},$$

where $\bar{J}$ is a skew-symmetric operator

$$\bar{J} = \bar{J}(\bar{u}) = \begin{bmatrix} 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & -\partial & -p & q \\ 0 & -2 & p & 0 & 2 \\ 2 & 0 & -q & -2 & 0 \end{bmatrix}. \tag{4.37}$$

It then follows that the enlarged hierarchy (4.36) possesses the following quasi-Hamiltonian structure:

$$\bar{u}_{t_m} = \bar{K}_m = \bar{J}\frac{\delta \bar{\mathcal{H}}_m}{\delta \bar{u}}, \qquad m \geqslant 0, \tag{4.38}$$

where $\bar{J}$ is the skew-symmetric operator defined by (4.37) and

$$\bar{\mathcal{H}}_m = \int \frac{2(a_{m+2} + e_{m+2})}{m+1}\, \mathrm{d}x, \qquad m \geqslant 0. \tag{4.39}$$

The first two nonlinear systems in (4.36) are

$$\begin{cases} p_{t_1} = p_x, q_{t_1} = q_x, v_{1,t_1} = \frac{1}{2}(pq)_x, \\ v_{2,t_1} = (p + v_2)_x + 2v_1 p - p^2 q, \\ v_{3,t_1} = (q + v_3)_x - 2v_1 q + pq^2, \end{cases} \tag{4.40}$$

$$\begin{cases} p_{t_2} = -\frac{1}{2}p_{xx} + p^2 q, q_{t_2} = \frac{1}{2}q_{xx} - pq^2, \\ v_{1,t_2} = \frac{1}{4}(pq_{xx} - p_{xx}q), \\ v_{2,t_2} = -\frac{1}{2}(p + v_2)_{xx} - (v_1 p)_x + p(pq + v_3 p + v_2 q) \\ \quad - v_1 p_x + v_2 pq - \frac{1}{2}p(pq_x - p_x q), \\ v_{3,t_2} = \frac{1}{2}(p + v_3)_{xx} - (v_1 q)_x - (pq + v_3 p + v_2 q)q \\ \quad - v_3 pq - v_1 q_x + \frac{1}{2}(pq_x - p_x q)q. \end{cases} \tag{4.41}$$

Case of $\delta_m = e_m$. Next let us choose

$$\delta_m = e_m, \qquad m \geqslant 0, \tag{4.42}$$

in (4.18). Then from (4.19), we have

$$v_{t_m} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}_{t_m} = S_m(u, v) = \begin{bmatrix} e_{m,x} \\ -2f_{m+1} - 2pe_m \\ 2g_{m+1} + 2qe_m \end{bmatrix}, \qquad m \geqslant 0,$$

where $v = (v_1, v_2, v_3)^T$. Moreover, the enlarged hierarchy of coupling systems for the AKNS hierarchy (4.5), defined by (4.20), reads

$$\bar{u}_{t_m} = \begin{bmatrix} u \\ v \end{bmatrix}_{t_m} = \bar{K}_m(\bar{u}) = \begin{bmatrix} K_m(u) \\ S_m(u, v) \end{bmatrix} = \begin{bmatrix} -2b_{m+1} \\ 2c_{m+1} \\ e_{m,x} \\ -2f_{m+1} - 2pe_m \\ 2g_{m+1} + 2qe_m \end{bmatrix}, \qquad m \geqslant 0. \tag{4.43}$$

It is easy to find that

$$\begin{bmatrix} -2b_{m+1} \\ 2c_{m+1} \\ e_{m,x} \\ -2f_{m+1} - 2pe_m \\ 2g_{m+1} + 2qe_m \end{bmatrix} = \bar{J} \begin{bmatrix} c_m + g_m \\ b_m + f_m \\ -2a_m \\ c_m \\ b_m \end{bmatrix},$$

where $\bar{J}$ is a skew-symmetric operator

$$\bar{J} = \bar{J}(\bar{u}) = \begin{bmatrix} 0 & 0 & -p & 0 & \partial \\ 0 & 0 & q & \partial & 0 \\ p & -q & \frac{1}{2}\partial & v_2 & -v_3 \\ 0 & \partial & -v_2 & 0 & -\partial + 2v_1 \\ \partial & 0 & v_3 & -\partial - 2v_1 & 0 \end{bmatrix}. \tag{4.44}$$

It then follows that the enlarged hierarchy (4.43) possesses the following quasi-Hamiltonian structure:

$$\bar{u}_{t_m} = \bar{K}_m = \bar{J}\frac{\delta\bar{\mathcal{H}}_m}{\delta\bar{u}}, \qquad m \geqslant 0, \tag{4.45}$$

where $\bar{J}$ is the skew-symmetric operator defined by (4.44) and

$$\bar{\mathcal{H}}_0 = \int 2v_1\,\mathrm{d}x, \qquad \bar{\mathcal{H}}_m = \int \frac{2(a_{m+1} + e_{m+1})}{m}\,\mathrm{d}x, \qquad m \geqslant 1. \tag{4.46}$$

The first two nonlinear systems in (4.43) are

$$\begin{cases} p_{t_1} = p_x, q_{t_1} = q_x, v_{1,t_1} = 0, \\ v_{2,t_1} = (p + v_2)_x + 2v_1 p, \\ v_{3,t_1} = (q + v_3)_x - 2v_1 q, \end{cases} \tag{4.47}$$

$$\begin{cases} p_{t_2} = -\frac{1}{2}p_{xx} + p^2 q, q_{t_2} = \frac{1}{2}q_{xx} - pq^2, \\ v_{1,t_2} = \frac{1}{2}(pq + v_3 p + v_2 q)_x, \\ v_{2,t_2} = -\frac{1}{2}(p + v_2)_{xx} - (v_1 p)_x + p(pq + v_3 p + v_2 q) \\ \quad - v_1 p_x + v_2 pq - p(pq + v_3 p + v_2 q), \\ v_{3,t_2} = \frac{1}{2}(p + v_3)_{xx} - (v_1 q)_x - (pq + v_3 p + v_2 q)q \\ \quad - v_3 pq - v_1 q_x + (pq + v_3 p + v_2 q)q. \end{cases} \tag{4.48}$$

The above two operators $\bar{J} = \bar{J}(\bar{u})$ do not satisfy the Jacobi identity and so they are not Hamiltonian. Nevertheless, since they are skew-symmetric, the corresponding integrable couplings are quasi-Hamiltonian.

4.5. *Conserved quantities*

Let $\{\bar{\mathcal{H}}_n\}_{n=0}^{\infty}$ be defined by (4.30), (4.39) or (4.46) in the above three examples. It then follows from the general theory on the Liouville integrability of zero curvature equations [7, 8] that

$$\{\bar{\mathcal{H}}_m, \bar{\mathcal{H}}_n\} := \int \frac{\delta\bar{\mathcal{H}}_m}{\delta\bar{u}}\bar{J}\frac{\delta\bar{\mathcal{H}}_n}{\delta\bar{u}}\,\mathrm{d}x = 0, \qquad m, n \geqslant 0.$$

Therefore, for the mth coupling system $\bar{u}_{t_m} = \bar{J}\frac{\delta\bar{\mathcal{H}}_m}{\delta\bar{u}}$ in each of the above three examples, we can compute

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t_m}\bar{\mathcal{H}}_n &= \int \frac{\delta\bar{\mathcal{H}}_n}{\delta\bar{u}}\bar{u}_{t_m}\,\mathrm{d}x \\ &= \int \frac{\delta\bar{\mathcal{H}}_n}{\delta\bar{u}}\bar{J}\frac{\delta\bar{\mathcal{H}}_m}{\delta\bar{u}}\,\mathrm{d}x \\ &= \{\bar{\mathcal{H}}_n, \bar{\mathcal{H}}_m\} = 0, \qquad n \geqslant 0.\end{aligned}$$

This implies that each Hamiltonian or quasi-Hamiltonian coupling system in the three hierarchies (4.26), (4.36) and (4.43), particularly, the coupling systems (4.27), (4.41) and (4.48), possesses infinitely many conserved quantities: $\{\bar{\mathcal{H}}_n\}_{n=0}^{\infty}$, defined by (4.30), (4.39) and (4.46), respectively.

5. Concluding remarks

The standard trace variational identity has been generalized to zero curvature equations associated with non-semi-simple Lie algebras or, equivalently, Lie algebras possessing degenerate Killing forms. The resulting generalized variational identity was applied to a class of semi-direct sums of Lie algebras in the AKNS case and furnished Hamiltonian and quasi-Hamiltonian structures of the associated integrable couplings. Three examples of integrable couplings for the AKNS hierarchy were presented: one Hamiltonian with four dependent variables and two quasi-Hamiltonian with five dependent variables.

For higher dimensions, we can combine zero curvature equations with Lax presentations, and both of them can be viewed as zero curvature conditions of connections. Starting with the parallel transport equation, Alvarez, Ferreira and Guillén have proposed an approach to construct conserved quantities and solutions in integrable theories in any dimension, on the basis of Virasoro-like algebras [14, 15]. In particular, the example of the $(2+1)$-dimensional CP^1 model [16] was discussed in detail and an infinite number of non-trivial local conserved quantities were explicitly constructed for a submodel of the $(2+1)$-dimensional CP^1 model [14, 15]. There also exist methods of Darboux transformations for constructing exact solutions to integrable equations in higher dimensions [17, 18]. In principle, all these integrable theories generalize the zero curvature condition connected with the time variable to nonlocal zero curvature conditions in spacetimes of higher dimensions, which can be associated with non-semi-simple Lie algebras. What is more, non-semi-simple Lie algebras were used to construct non-isospectral flows and τ-symmetries of time-independent soliton equations (see, for example, [19–21] for the continuous case and [22, 21] for the discrete case) and time-dependent soliton equations (see, for example, [23–26]). Our theory in this paper provides a method to establish local Hamiltonian and quasi-Hamiltonian integrable couplings by using semidirect sums of Lie algebras, for given integrable equations in $1+1$ dimensions.

We finally point out that our two examples with five dependent variables in the previous section only possess quasi-Hamiltonian structures. It is very intriguing to us how to construct Hamiltonian integrable coupling structures with five dependent variables for the AKNS hierarchy.

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